Graphing Lines and Systems of Equations

Finite Math

9 October 2018

Quiz

What is your favorite city you've ever visited?

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What is your favorite city you've ever visited? Mine:

Definition

Definition (Line)

A line is the graph of an equation of the form

$$Ax + By = C$$

where not both of A and B are equal to zero (i.e., if $A = 0$, then $B \neq 0$ and vice-versa).

Graphing Lines

There are two common ways of graphing lines: by **finding intercepts** and by **using the slope and a point.** We will focus on the method of finding intercepts here in the notes. You can read about using the slope to graph a line in the textbook.

Finding Intercepts

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A point of the form $(a, 0)$ *on a line is called an* x*-intercept and a point of the form* $(0, b)$ *is called a* y*-intercept.*

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Every line will have at least one intercept, but most have two. There are three special cases in which the line has only one intercept: if $A = 0$, $B = 0$, or $C = 0$. We will return to these special cases in a little bit.

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Assume the line $Ax + By = C$ has both an x- and y- intercept, we find them as follows:

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- To find the y-intercept, we set $x = 0$ in the equation of the line and solve for y. Symbolically, this means that
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To graph a line using intercepts, we plot the two intercepts in the *xy*-plane, and draw a line through the points:

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Example

Graph the line $4x - 3y = 12$ *using intercepts.*

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If $C = 0$, you'll find that solving for the x-intercept as above gives $(0, 0)$ and solving for the y-intercept also gives $(0, 0)$.

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Graphing a Line Through the Origin

Example

Graph the line $2x + 3y = 0$.

Horizontal and Vertical Lines

The cases when $A = 0$ or $B = 0$ in $Ax + By = C$ correspond to horizontal and vertical lines, respectively.

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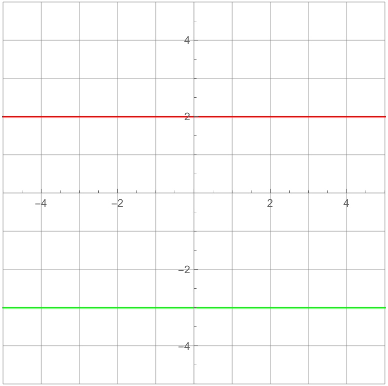
If $A = 0$, we end up with the line $y = \dfrac{C}{B}$, which is a horizontal line where every y-value is $\dfrac{C}{B}$. A special one of these is when C is also zero so we get the equation $y = 0$. The graph of this line is the x-axis.

Horizontal and Vertical Lines

Here are the graphs of $y = 2$ (red) and $y = -3$ (green).



Horizontal and Vertical Lines

The cases when $A = 0$ or $B = 0$ in $Ax + By = C$ correspond to horizontal and vertical lines, respectively. If $B = 0$, we end up with the line $x = \frac{C}{A}$, which is a vertical line where every x-value is $\frac{C}{A}$. A special one of these is when C is also zero so we get the equation $x = 0$. The graph of this line is the y-axis.

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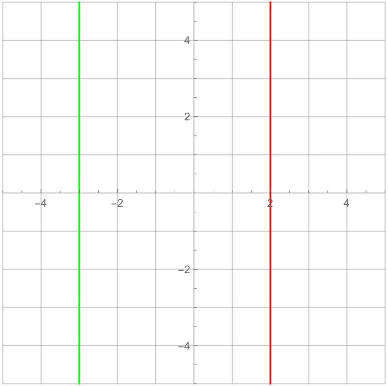

Now You Try It!

Example

Graph the following lines:

(a)
$$2x - y = 3$$

(b)
$$2x + 4y = 8$$

(c)
$$3x - 2y = 0$$

(d)
$$6x = 18$$

Motivating Example

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Package 1: 2 adult tickets and 1 child ticket for \$32
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Based off of this information, can we figure how much the adult and child ticket discount prices are?

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This is a system of two linear equations in two variables. To find the answer, we need to find a pair of numbers (A, C) which satisfy *both* equations simultaneously.

Definition

Definition (System of Two Linear Equations in Two Variables)

Given the linear system

$$\begin{array}{ccccc} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

where a*,* b*,* c*,* d*,* h*, and* k *are real constants, a pair of numbers* $x = x_0$ *and* $y = y_0$ *(often written as an ordered pair* (x_0, y_0)*) is a solution of this system if each equation is satisfied by the pair. The set of all such ordered pairs is called the solution set for the system. To solve a system is to find its solution set.*

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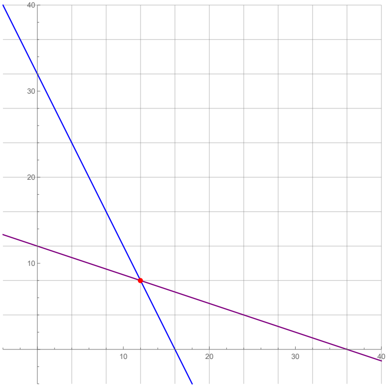
There are a few ways we can go about solving this: *graphically*, using *substitution*, and *elimination by addition*.

Solving by Graphing

To solve this problem by graphing, we simply graph the two equations in the system, then find the intersection. Since we're relying on a graph to find this point, we need to check our solution in the equations of the system.



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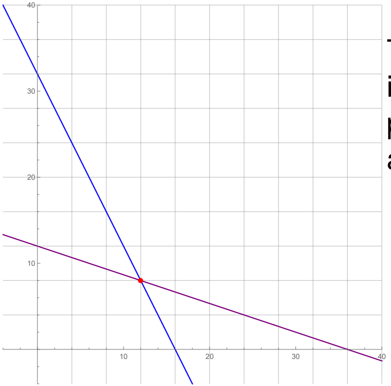

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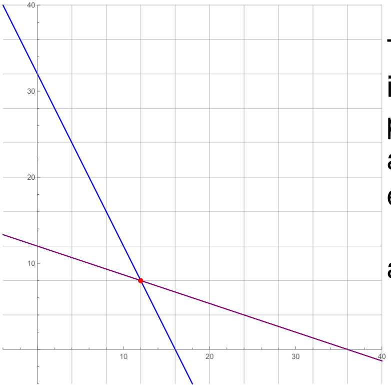

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This verifies the solution.

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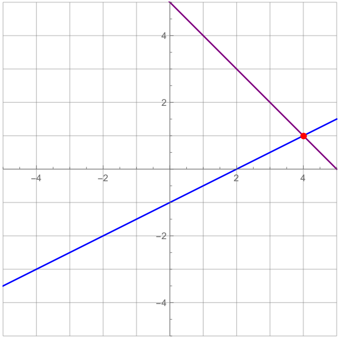

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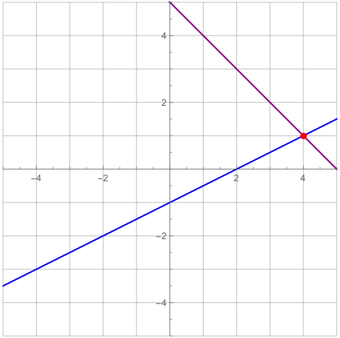

In this case, like before, we see only the *one solution* at $(4, 1)$. (You should check this in the system!)

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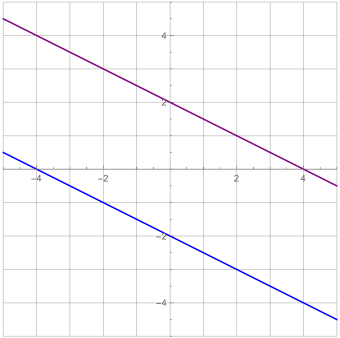

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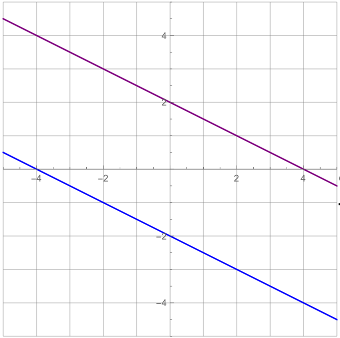

In this case, the lines are parallel and so they never intersect. In this case, there is *no solution*.

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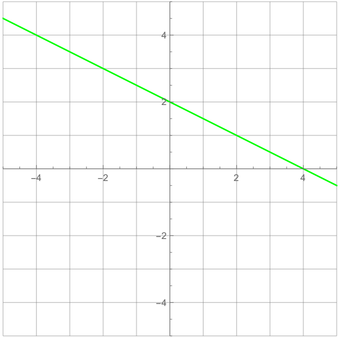

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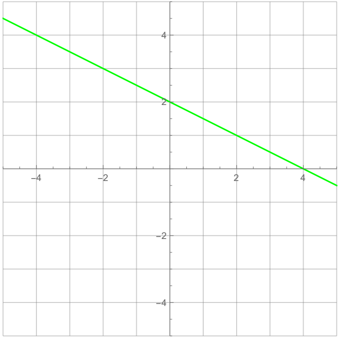

Here, both of the lines are exactly the same.
In this case, there is an infinite number of solutions.

Terminology

Definition

A system of linear equations is called consistent if it has one or more solutions and inconsistent if it has no solutions. Further, a consistent system is called independent if it has exactly one solution (called the unique solution) and is called dependent if it has more than one solution. Two systems of equations are called equivalent if they have the same solution set.

Theorem

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The linear system

$$\begin{array}{rcrcl} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

must have

1. *Exactly one solution (consistentent and independent).*
2. *No solution (inconsistent).*
3. *Infinitely many solutions (consistent and dependent).*

There are no other possibilities.

Now You Try It!

Example

Solve the following systems of equations using the graphing method. Determine whether there is one solution, no solutions, or infinitely many solutions. If there is one solution, give the solution.

(a)

$$\begin{array}{rcrcr} x & + & y & = & 4 \\ 2x & - & y & = & 5 \end{array}$$

(b)

$$\begin{array}{rcrcr} 6x & - & 3y & = & 9 \\ 2x & - & y & = & 3 \end{array}$$

(c)

$$\begin{array}{rcrcr} 2x & - & y & = & 4 \\ 6x & - & 3y & = & -18 \end{array}$$

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When solving a system by substitution, we solve for one of the variables in one of the equations, then plug that variable into the other equation.

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Example

Solve the following system using substitution

$$\begin{array}{rcrcr} 2x & - & y & = & 3 \\ x & + & 2y & = & 14 \end{array}$$

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Example

Solve the following system using substitution

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Solution

$x = -2, y = 2$

Solving Using Elimination

We now turn to a method that, unlike graphing and substitution, is generalizable to systems with more than two variables easily. There are a set of rules to follow when doing this

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Theorem

A system of linear equations is transformed into an equivalent system if

1. *two equations are interchanged*
2. *an equation is multiplied by a nonzero constant*
3. *a constant multiple of one equation is added to another equation.*

Solving Using Elimination

Example

Solve the following system using elimination

$$\begin{array}{rcrcr} 3x & - & 2y & = & 8 \\ 2x & + & 5y & = & -1 \end{array}$$

Now You Try It!

Example

Solve the system using elimination

$$\begin{array}{rcrcl} 5x & - & 2y & = & 12 \\ 2x & + & 3y & = & 1 \end{array}$$

Now You Try It!

Example

Solve the system using elimination

$$\begin{array}{rcrcl} 5x & - & 2y & = & 12 \\ 2x & + & 3y & = & 1 \end{array}$$

Solution

$x = 2, y = -1$